Seifert Surfaces

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Introduction

Knot theory is the study of mathematical elements that represent closed knotted loops.

Definitions:

Knot - *A knot* is a closed curve in three-dimensional space that does not intersect itself and is fully embedded within that space.

Link - Informally, *a link* is a collection of knots that are entangled with each other in three-dimensional space.

Examples of Knots

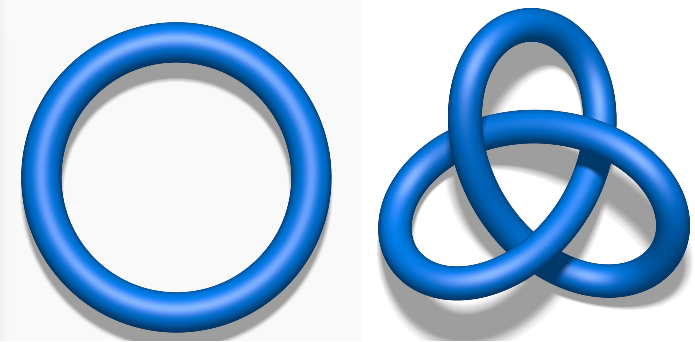
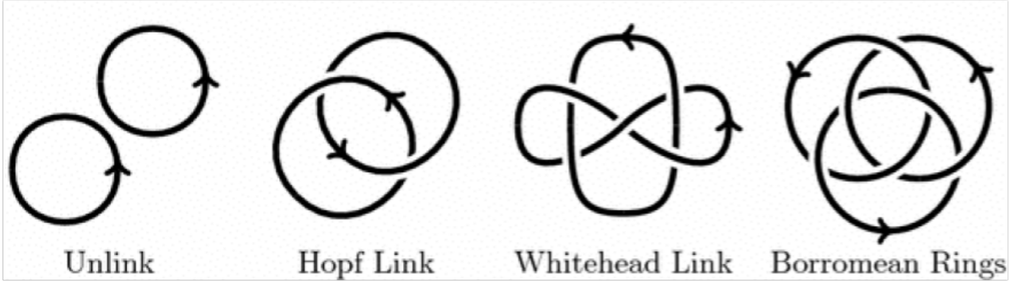


Figure: The simplest knot and the second simplest

Examples of Links



Knot Composition

Definition: *Knot composition*, also known as the connected sum of knots, is a method of combining two knots to form a new knot.

Process:

1. Cut each knot at any point.
2. Join the boundaries of the cuts
3. There are no orientations that we need to keep consistent, so all compositions result in the same knot

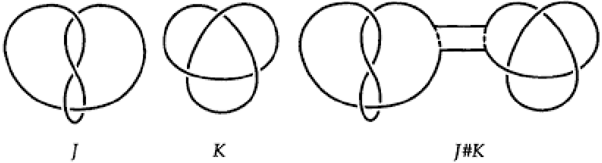


Figure: Combining knots J and K

Prime Knots

Prime Knots:

- A knot is prime if it cannot be represented as a connected sum of two nontrivial knots.
- Any knot not prime is composite.

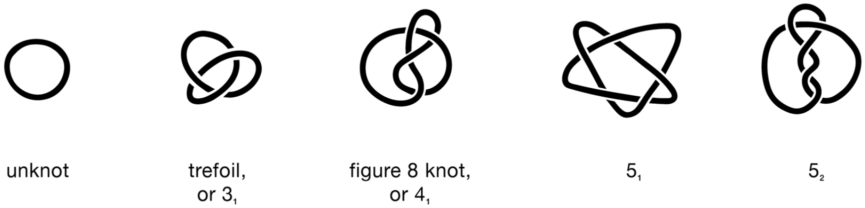


Figure: Examples of Prime Knots

Surfaces

Definition: *A surface* is a shape such that there can be any point with a disk surrounding and containing it.

For example, it is like the glaze on a donut or the paint surrounding a mug.

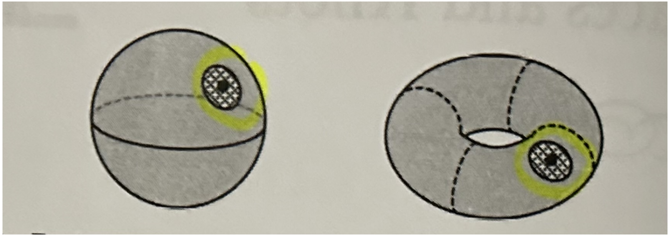


Figure: A sphere and a torus

Boundary

Definition A surface with boundary is a surface where at least one open disk, known as a boundary component, has been removed, in dimension 2. This transforms it from a closed surface to one with an exposed edge.

Example: Consider a torus - If a disk is removed, the boundary consists of the rim surrounding the removed disk.

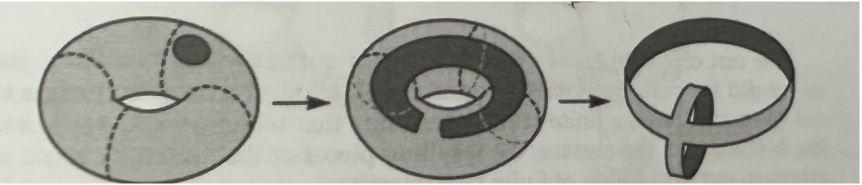
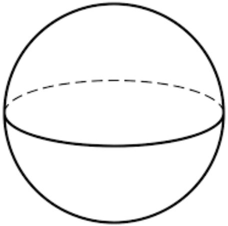
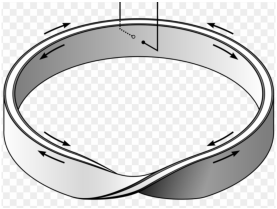


Figure: Torus with one boundary

Orientability

- **Orientability:** A surface is orientable if it has two distinct sides.
- **Two-sidedness:** Imagine painting the two sides of a surface different colors, like red and blue.
- **No Mixing:** If the red paint never touches the blue paint except along the boundary, the surface is orientable.
- **Examples:** Sphere and the torus
- **Non-orientable Surfaces:** The Möbius strip is non-orientable, as it only has one side.





Orientability (Continued)

Figure: A two-twist band is orientable

Seifert Surfaces

Seifert surfaces are two-sided surfaces embedded in three-dimensional space whose boundary is a knot or a link.

They provide a powerful tool for understanding knot structures and properties.

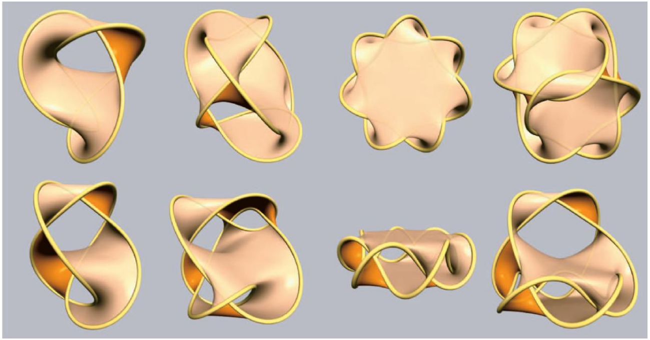


Figure: Seifert Surfaces

Seifert's Algorithm

Below Seifert's Algorithm. However on the next few slides we will take a closer look at each step.

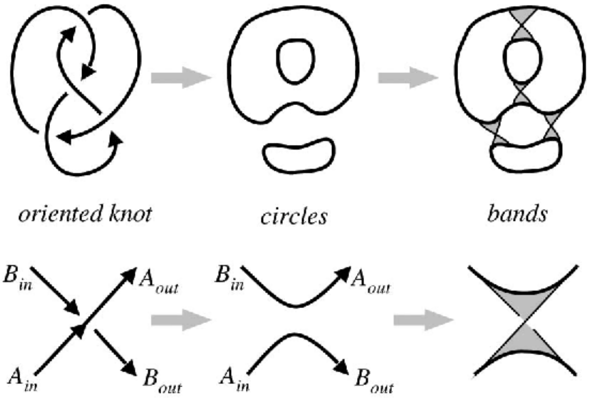
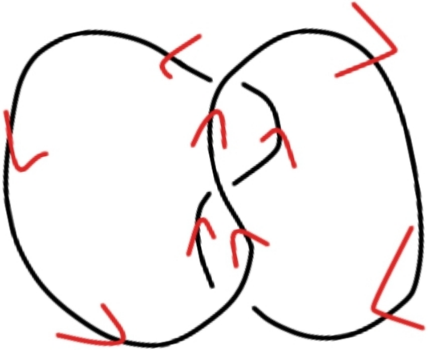


Figure: Seifert's Algorithm

Step 1

Fix an Orientation

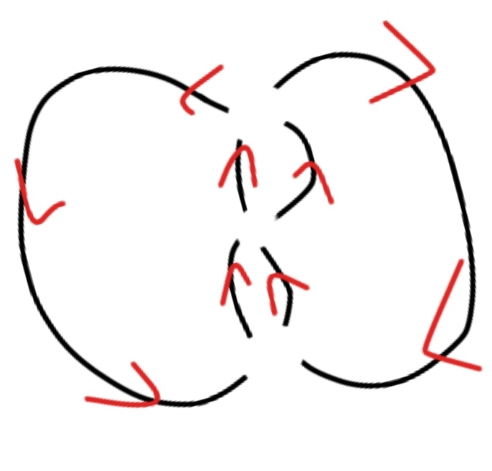
- Choose a consistent direction for the knot.
- This helps in resolving crossings systematically.
- Ensures uniformity in the process.



Step 2, Part 1

Resolving Crossings

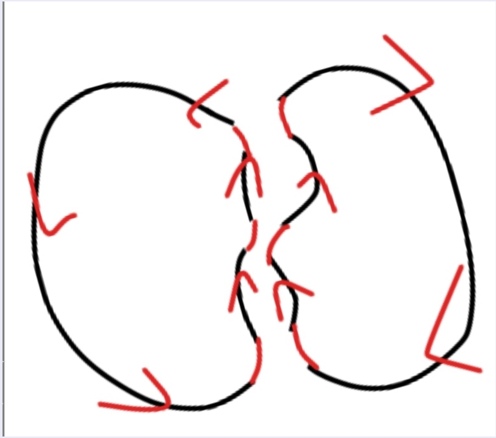
- Untangle the knot by adjusting each crossing.



Step 2, Part 2

Resolving Crossings

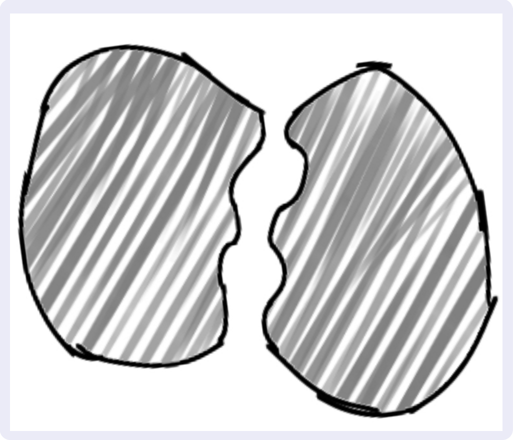
- Align crossings with the chosen orientation.
- Simplifies the knot into non-overlapping strands.



Step 3

Placing Disks

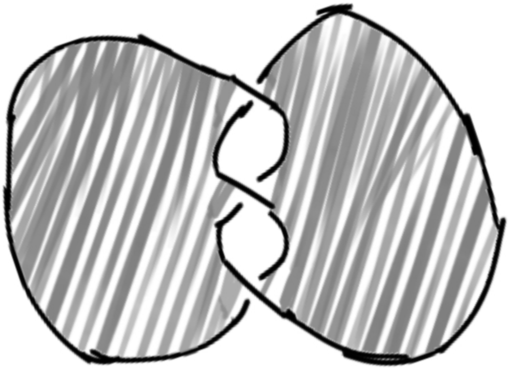
- Place disks inside the regions created by resolving crossings.
- These disks define necessary boundaries.
- Each bounded region is enclosed by a disk.



Step 4

Adding Bands

- Connect disks with bands at crossings.
- Bands act as bridges, linking regions together.
- Ensure the Seifert surface is connected.
- The way the band is placed depends on the direction of the twists of the crossing.



Genus of a Knot

Definition:

- The genus of a knot is the minimum genus (number of "holes") of any Seifert surface bounding the knot.
- It is an important invariant in knot theory.

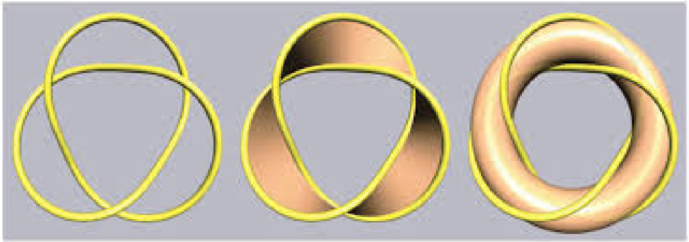


Figure: The Trefoil knot has a genus of 1

Genus of a Knot (Continued)

Key Points:

- The Seifert surface itself is not unique.
- Different Seifert surfaces for the same knot may have different appearances, changing the genus.
- This is why we take the minimum of any such surface as the genus of the knot.

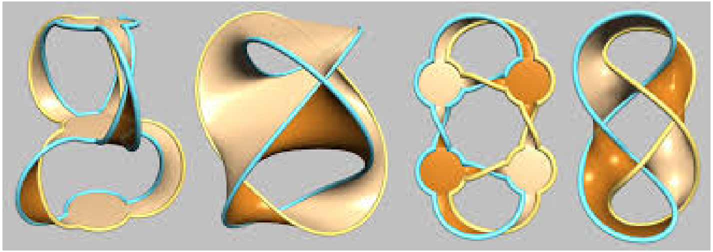


Figure: Different configurations for Whitehead link

Genus and Connected Sum

Genus formula - For knots K and J, the genus of their connected sum satisfies:

$$g(K\#J) \leq g(K) + g(J)$$

The genus of the connected sum is always less than or equal to the sum of the genera of the individual knots.

Explanation:

- The inequality holds because the connected sum of two knots connects their Seifert surfaces.
- If we use the minimal Seifert surfaces of each knot, the genus of the combined surface is at most the sum of the individual genera.

Genus and Connected Sum (Continued)

Theorem:- For knots K and J, the genus of their connected sum satisfies:

$$g(K\#J) = g(K) + g(J)$$

The genus of the connected sum is exactly the sum of the genera of the individual knots.

Explanation:

- This theorem holds because when you form a connected sum of two knots, you essentially connect their minimal Seifert surfaces together.

Can the Sum of two knots be the unknot?

If

$$K\#J = \text{unknot}$$

Then

$$g(K\#J) = g(O)$$

So

$$g(K) = g(J) = 0$$

Proving they were both the unknot to begin with

Genus and Prime Knots

Similarly that if

$$g(K) = 1$$

Then if

$$K = J_1 \# J_2$$

We have

$$g(J_1 \# J_2) = 1$$

So one of the two had

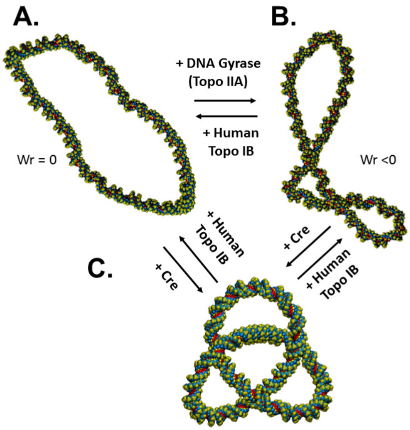
$$g(J_i) = 0$$

This conveys one of the two knots has to be the unknot, so K is prime.

Real World Applications Of Knot Theory

DNA Topology

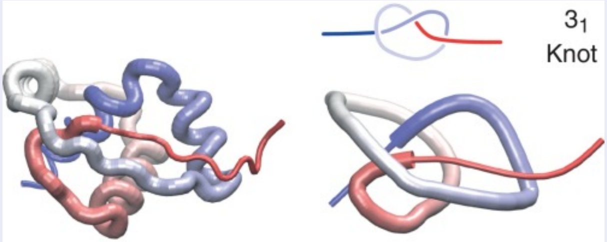
Understanding how DNA strands can be knotted or linked is crucial for processes like replication, transcription, and recombination.



Real World Applications (Continued)

Protein Folding

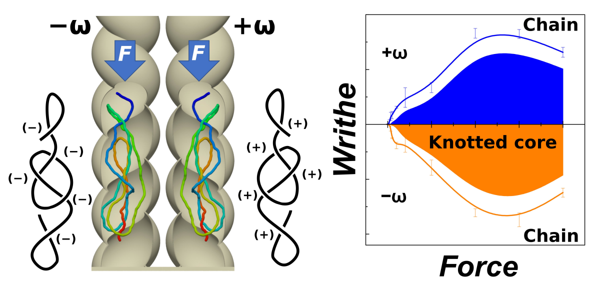
Knot theory contributes to understanding protein folding and molecular biology, where proteins often adopt complex knotted configurations.



Real World Applications (Final)

Biomaterial Engineering

Supports engineers in understanding how the topology of biomaterials influences their mechanical properties, such as flexibility, strength, and elasticity. They can tailor their properties to specific applications.



References

Adams, C.C. (2004) The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots. American Mathematical Society, Providence.

Image Sources

- Mathworld: Knot Sum
- Wikipedia: Knot Table
- Visualization of Seifert Surfaces Paper
- Google Image: Knot
- Google Image: Knot 2

References Continued

Image Sources

- ResearchGate: Knot and Link Diagrams
- Open University: Knot Diagram
- Semantic Scholar: Seifert Surfaces
- Research Features: DNA Topology
- ScienceDirect: Knot Theory in Biology
- MDPI: Polymers

End

Thank you for your time and special thanks to the Primes Circle coordinators, Mary and Marisa!

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